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Aerodynamics of Slender Lifting Surface in Accelerated Flight

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I. Introduction

THERE are many researches into the aerodynamics of lifting surfaces in unsteady motion. Most of these, however, concern out-of-plane motions at a uniform flight speed, with few studies for nonuniform flight speeds.[1,2,3] These problems have become increasingly more important, because recent airplanes, such as STOL or supersonic, can experience considerable acceleration during the take-off climb or landing approach. More precise estimation of take-off or landing distance would require some aerodynamic theories for accelerated flight.

In Ref. 4, a fundamental formulation was made for wings in nonuniform motion in an inviscid incompressible flow. A Fourier transform for the in-plane coordinate variables was used. In Ref. 5, application was made to two-dimensional airfoils with a sinusoidally pulsating speed. A significant result was the fact that the difference between unsteady and quasi-steady lifts is significant, as was also shown by the previous writers.[2,3] In general, the lift of a wing depends on the complete history of its motion, and not just on the instantaneous acceleration.

In this paper, we treat the problem of slender wings in accelerated motion. This problem also is solvable and represents the low aspect ratio limit of wing theory opposite that of the two-dimensional problem. An interesting result has been obtained in that the lift force for slender wings is shown to depend only on the instantaneous acceleration and not on the history of its motion. An acceleration (deceleration) increases (decreases) the lift from that for the uniform flight speed.

II. General Formulation

In this section, a general formulation is given for the case of a wing in nonuniform motion (flight speed) in an inviscid fluid. The analysis employs a moving axes system fixed to the wing. According to Reissner,[6] the unsteady linear equations of aerodynamics expressed in a moving axes system can be written as follows:

$$u_i' = \nabla\phi \tag{1}$$

$$p_i/\rho = (U+\Omega\times r)\cdot\nabla\phi - \partial\phi/\partial t \tag{2}$$

$$\nabla^2\phi - (1/a_0^2)[\partial/\partial t - (U+\Omega\times r)\cdot\nabla]^2\phi = 0 \tag{3}$$

$$(\partial\phi/\partial n)\,|\nabla F_{Lp}| = (U+\Omega\times r)\cdot\nabla f_L - \partial f_L/\partial t \tag{4}$$

Here u_i' denotes the velocity of a fluid particle relative to the inertia system (fixed to the earth), the subscript i denotes disturbances, ϕ is the velocity potential, $r=ix+jy+kz$ denotes the position of a fluid particle relative to the moving axes system, p and ρ denote pressure and mass density of the air, respectively, U and Ω denote the linear and angular velocities of the moving axes system relative to the inertia system, respectively, a_0 is the sonic speed in the undisturbed condition, and t is time. An equation for the wing surfaces may be written as

$$F_L(x,y,z,t) = F_{Lp} - f_L = 0 \tag{5}$$

where F_{Lp} concerns the fundamental wing surface making no disturbances, f_L concerns a small out-of-plane displacement, ∇ stands for $i\partial/\partial x + j\partial/\partial y + k\partial/\partial z$. Equations (1-4) are valid for subsonic, transonic, and supersonic regions, provided that the fluid is inviscid and the disturbances are small.

We also make the following three additional assumptions:

1) The wing makes only a straight translation in still air in the negative-x direction

$$U = -U(t)\,i \qquad u_0' = 0 \quad \Omega = 0 \tag{6}$$

2) The wing lies nearly in the xy plane

$$F_{Lp} - f_L = z - f_L(x,y,t) = 0 \tag{7}$$

3) The flight velocity $U(t)$ is nonzero and positive. Then we may replace time, t with the flight distance τ, defined as follows:

$$\tau = \int_0^t U(t_I)\,\mathrm{d}t_I \tag{8}$$

Thus,

$$\partial/\partial t + U(t)\,\partial/\partial x = \tilde U(\tau)\,(\partial/\partial x + \partial/\partial\tau) \qquad U(t) \equiv \tilde U(\tau) \tag{9}$$

and Eqs. (1-4) reduce to

$$u_i' = \nabla\phi \tag{10}$$

$$p_i/\rho = -\tilde U(\tau)\,(\partial/\partial x + \partial/\partial\tau)\phi \tag{11}$$

$$\nabla^2\phi = (1/a_0^2)[\tilde U(t)\,(\partial/\partial x + \partial/\partial\tau)]^2\phi \tag{12}$$

$$\partial\phi/\partial z = +\tilde U(\tau)\,(\partial/\partial x + \partial/\partial\tau) f_L \tag{13}$$

The governing differential equation, Eq. (12), may be rewritten in a more explicit form:

$$\begin{aligned}[1-\tilde M_0^2(\tau)]\phi_{xx} + \phi_{yy} + \phi_{zz} \\ = \tilde M_0^2(\tau)[2\phi_{x\tau} + \phi_{\tau\tau} + (\tilde U'/\tilde U)(\phi_x + \phi_\tau)]\end{aligned} \tag{14}$$

where $M_0(t) \equiv \tilde M_0(\tau)$ denotes instantaneous flight Mach number and $\tilde U'(\tau) = \mathrm{d}\tilde U/\mathrm{d}\tau$.

Although Eq. (14) is linear in the velocity potential ϕ, it contains variable coefficients dependent on τ. Hence, we cannot use the concept of frequency response, and analytically solvable cases may be rare. Two simple cases exist, one for two-dimensional airfoils, the other for slender wings. The latter is discussed in the following sections.

III. Slender Wing in Accelerated Flight

Let us denote l_x, l_y, and l_z as the reference lengths in the x,y,z directions, assuming $l_y = l_z \equiv l$. We also denote t_I and τ_I as the reference time and flight distance, and assume $\tau_I = Ut_I$. In the special case when the flight speed is simply harmonic in time with period T_I, we may take $t_I = T_I$. Using the con-

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ventional assumptions for unsteady slender wings,

$$l/l_x \ll l \quad \text{and} \quad l/Ut_l \ll l \tag{15}$$

Eq. (14) reduces to the two-dimensional Laplace equation in the cross flow plane. Thus, the problem may be recast in the following form:

Govering equation

$$\partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0 \tag{16}$$

Boundary condition on the wing surface

$$\partial\phi/\partial z = +\tilde{U}(\tau)(\partial/\partial x + \partial/\partial\tau)f_L \tag{17}$$

Disturbance pressure

$$p_i = -\rho\tilde{U}(\tau)(\partial/\partial x + \partial/\partial\tau)\phi \tag{18}$$

The second assumption of Eq. (15) has been used for an approximate treatment of out-of-plane motions of slender wings[7] which require less rapid oscillation. Here we adopt this assumption for in-plane nonuniform motions, too.

Let $b(x)$ denote the local semispan of a slender wing, at a streamwise position x; the origin being at the front end (apex) of the wing. In order to solve the two-dimensional Laplace equation with a given boundary condition on a segment, we introduce the complex plane $\zeta = y + iz$ and the complex velocity potential $F(\zeta)$ which, in turn, defines the conjugate complex velocity $q(\zeta) = \mathrm{d}F/\mathrm{d}\zeta$. Then, we have the following problem: Obtain a sectionally analytic function $q(\zeta)$ that satisfies the boundary condition

$$q^+(y) + q^-(y) = -2iw(y) \quad -b \leqq y \leqq b, \quad z = 0 \tag{19}$$

which may have an integrable singularity at both ends $\zeta = \pm b$ and which vanishes at infinity.

This is a well-known nonhomogeneous Hilbert problem.[8] Once the function $q(\zeta)$ is obtained, integration of it by ζ gives $F(\zeta)$. The real part of $F(\zeta)$ is just the velocity potential ϕ being desired. Now $q(\zeta)$ belongs to the class h_0 because $q(\pm b) = \infty$. According to the formal procedure, we have

$$q(\zeta) = \frac{1}{\pi\sqrt{b^2-\zeta^2}} \int_{-b}^{b} \frac{w(y)\sqrt{b^2-y^2}}{\zeta - y}\,\mathrm{d}y \tag{20}$$

Integration of Eq. (20) by ζ gives

$$F(\zeta) = \frac{1}{\pi}\int_{-b}^{b} \sqrt{b^2-y_1^2}\,w(y_1)\,\mathrm{d}y_1 \int^{\zeta} \frac{\mathrm{d}\zeta_1}{\sqrt{b^2-\zeta_1^2}(\zeta - y)}$$

Let us define the integral constant so that $F(\zeta)$ vanishes at $\zeta = \pm b$. Then we have

$$F(\zeta) = \frac{1}{2\pi}\int_{-b}^{b} L(\zeta, y_1, b)\,w(y_1)\,\mathrm{d}y_1 \tag{22}$$

where

$$L(\zeta, y_1, b) \equiv \ln\frac{(\zeta - y_1)^2 + (\sqrt{b^2-\zeta^2} - \sqrt{b^2-y_1^2})^2}{(\zeta - y_1)^2 + (\sqrt{b^2-\zeta^2} + \sqrt{b^2-y_1^2})^2} \tag{23}$$

Thus we have the velocity potential ϕ at the wing surfaces, $-b \leqq y \leqq b$, $z = 0$ as follows:

$$\phi(x,y) = \frac{1}{2\pi}\int_{-b(x)}^{b(x)} w(y_1)\,L(y, y_1, b(x))\,\mathrm{d}y_1 \tag{24}$$

where x affects ϕ only as a parameter. The disturbance pressure on the wing surfaces is given by using Eqs. (18) and (24);

$$\Delta p = \frac{1}{\pi}\rho\tilde{U}(\tau)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial\tau}\right)\int_{-b(x)}^{b(x)} w(y_1)\,L(y, y_1, b)\,\mathrm{d}y_1 \tag{25}$$

Making use of

$$w(y) = +\tilde{U}(\tau)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial\tau}\right)f_L \tag{26}$$

we then have

$$\Delta p = \Delta p_1 + \Delta p_2 \tag{27}$$

where

$$\frac{\Delta p_1}{\rho U^2} = +\frac{1}{\pi}\int_{-b(x)}^{b(x)} \left\{L(y, y_1, b)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial\tau}\right)^2 f_L + \left(\frac{\partial f_L}{\partial x} + \frac{\partial f_L}{\partial\tau}\right)\frac{\partial L}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}x}\right\}\mathrm{d}y_1 \tag{28}$$

$$\frac{\Delta p_2}{\rho U^2} = +\frac{1}{\pi}\frac{\tilde{U}'}{U}\int_{-b(x)}^{b(x)} L(y, y_1, b)\left(\frac{\partial f_L}{\partial x} + \frac{\partial f_L}{\partial\tau}\right)\mathrm{d}y_1 \tag{29}$$

The former, Δp_1, is produced by the out-of-plane motions of a slender wing, while the latter, Δp_2, is due to the in-plane motions. It is noteworthy that both Δp_1 and Δp_2 depend on the instantaneous condition and not on the history of the motion. This is a unique property for slender wings, which has been known for the out-of-plane contribution Δp_1. The other one, Δp_2, is a new term, which is proportional to the instantaneous acceleration $\dot{U}$. As far as the present author is aware, this interesting fact is new and has not been pointed out by previous investigators.

Another interesting question is whether or not an acceleration produces a contribution to the quasisteady lift value. For this purpose, we investigate the simplest case of a flat slender delta wing having a steady incidence α

$$f_L = -\alpha x \tag{30}$$

Then Eq. (29) gives

$$\frac{\Delta p_2}{\rho U^2} = +2\alpha\frac{\tilde{U}'}{U}\sqrt{b^2 - y^2} \tag{31}$$

and the corresponding lift force L_{acc} is

$$L_{\text{acc}}/\rho U^2 = +(\pi/3)\alpha(\tilde{U}'/U)C_0 b_0^2 \tag{32}$$

where C_0 and b_0 denote the maximum chord length and the semispan. On the other hand, the quasisteady lift L_{qs} at velocity U is written as

$$L_{qs} = \pi\alpha\rho U^2 b_0^2 \tag{33}$$

Thus, we have

$$L_{\text{acc}}/L_{qs} = +\tfrac{1}{3}(\dot{U}C_0/U^2) \tag{34}$$

which shows that an acceleration (deceleration) increases (decreases) the lift force. Dotted quantity means the derivative by time t. Moreover Eq. (34) suggests that the appropriate nondimensional (or reduced) acceleration should be defined as

$$A \equiv \dot{U}C_0/U^2 \tag{35}$$

which includes the longitudinal reference length C_0, as expected. The physical meanings of Eq. (34) might be inferred in the following way: if the angle-of-attack is positive, a forward acceleration of a wing surface brings the downward component normal to the wing surface. Then, according to the theory of out-of-plane wing motions, a lift due to virtual mass effect is produced. It is easy to show that this reasoning is valid for a flat delta wing. In fact, from Δp_l we have the following part

$$\Delta p_l = 2\rho(-f_{Ltt})\sqrt{b^2-y^2} \tag{36}$$

which is equivalent to Eq. (31) if we replace $\alpha\dot{U}=\alpha U\tilde{U}'$ by $-f_{Ltt}$.

IV. Conclusions

The lift force of a slender wing is analytically determined for nonuniform flight speed. The fluid is assumed to be inviscid. The flight speed may be either subsonic, transonic, or supersonic. It is found that the lift depends on the instantaneous acceleration and not on the history of the wing motion. An acceleration (or deceleration) increases (decreases) the lift force from that of the steady-state. The reason for this may be attributed to the component of a forward acceleration of a wing normal to the surface.

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High-Frequency Subsonic Flow Past a Pulsating Thin Airfoil II: Gust-Type Upwash

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Introduction

IN Ref. 1, the solution was given for the high-frequency subsonic potential flow past a thin airfoil pulsating harmonically in time. This closed-form high-frequency approximation is an example of a symmetric (nonlifting) disturbance to a subsonic stream and, as such, complements the existing solutions for lifting problems (see, for example, Ref. 2). The possible applications are in aeroacoustics. In this Note, the results are extended to treat an example where the scale of the streamwise variation of the airfoil shape is comparable to the acoustic wavelength; the interesting case of a Sears gust-type upwash is studied.

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Analysis

The first-order perturbation velocity potential for two-dimensional flow of a uniform stream of speed U, Mach number $M(\beta^2 \equiv 1-M^2)$, and sound speed a past a pulsating airfoil with surface

$$y = \pm\epsilon g(x)\exp(i\omega t); \quad -\ell \leq x \leq \ell \tag{1}$$

is given in Ref. 1 as

$$\epsilon\phi^{(I)}(x,y)\exp(i\omega t) = i\epsilon(2\beta)^{-1}\int_{-\ell}^{\ell}[i\omega g(\xi)+Ug'(\xi)] \times\exp\left[\frac{i\omega M(x-\xi)}{a\beta^2}\right]H_0^{(2)}\left\{\frac{\omega[(x-\xi)^2+\beta^2y^2]^{1/2}}{a\beta^2}\right\}\mathrm{d}\xi \times\exp(i\omega t) \tag{2}$$

In Ref. 1, the asymptotic expansion of Eq. (2) was given for $\omega\to\infty$ with $g(x)$ and $g'(x)$ of $O(1)$. Consider pulsation where $g(x)$ varies on the scale of the acoustic wavelength $\lambda = 2\pi a\omega^{-1}$. It is of theoretical interest to choose the upwash to be the symmetrical counterpart of that for the Sears-type compressible gust:

$$\phi_y^{(I)}(x,0\pm) = \pm w_0 e^{-i\omega x/U}; \quad -\ell\leq x\leq\ell \tag{3}$$

This upwash is realized if the airfoil shape function is chosen to be

$$g(x) = w_0 U^{-1} x e^{-i\omega x/U} \tag{4}$$

Substitution into Eqs. (2) and (4) yields

$$\phi^{(I)} = iw_0(2\beta)^{-1}\exp\left(-\frac{i\omega x}{U}\right)\int_{-\ell}^{\ell}\exp\left[\frac{i\omega(x-\xi)}{Ma\beta^2}\right] \times H_0^{(2)}\left\{\frac{\omega[(x-\xi)^2+\beta^2y^2]^{1/2}}{a\beta^2}\right\}\mathrm{d}\xi \tag{5}$$

There is a contribution to the asymptotic value of the integral from an interior critical point and the two end points. The following results will be invalid for $|x+\ell| = O(\lambda)$ and $|x-\ell| = O(\lambda)$ due to the coalescing of the interior critical point with one of the end points. An appropriate scale for y is λ, and therefore a solution is sought in the outer region $y = O(\lambda)$. In the evaluation of the integral in Eq. (5), the variable $Y = \omega y$, which is $O(1)$, is used.

The solution is written as

$$\phi^{(I)} = \phi^{(II)} + \phi^{(IL)} + \phi^{(IT)} \tag{6}$$

where the superscripts I, L, and T represent the interior critical point, forward end point, and rear end point. The interior critical point is a zero at $x=\xi$. Its contribution is obtained by using Eq. (6.4.9) in Ref. 3 to yield

$$\phi^{(II)} = \begin{cases} -Uw_0\omega^{-1}\exp[-\omega(ix+|y|)U^{-1}]; & |x|<\ell \\ 0; & |x|>\ell \end{cases} \tag{7}$$